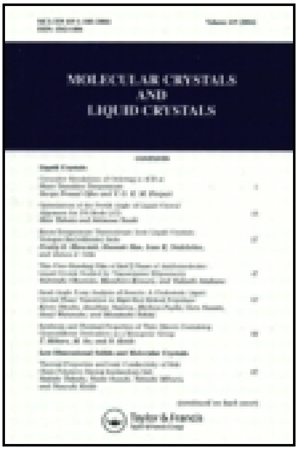

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Axisymmetrical Flow of a Nematic Liquid Crystal around a Sphere

H. Kneppe [a], F. Schneider [a] & B. Schwesinger [b]
[a] Physikalische Chemie, Universitat Siegen, 5900, Siegen, F. R. Germany
[b] Theoretische Physik, Universität Siegen, 5900, Siegen, F. R. Germany
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H. KNEPPE and F. SCHNEIDER
Physikalische Chemie, Universität Siegen, 5900 Siegen, F. R. Germany

and

B. SCHWESINGER
Theoretische Physik, Universität Siegen, 5900 Siegen, F. R. Germany

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The axisymmetrical steady flow of an incompressible nematic liquid crystal around a sphere is studied on the basis of the Leslie-Ericksen equations. We assume low Reynolds numbers and a fixed director orientation parallel to the moving sphere (or the moving liquid crystal at infinite distance). The influence of the Leslie coefficient α_1 is neglected. We find a series representation for the streamfunction and the pressure. The force on the sphere, which only depends on the viscosity coefficients η_1 and η_2 but not on η_3, can be represented by a simple formula. Stream line patterns and the pressure distribution on the sphere are presented for several viscosity coefficient ratios. The determination of viscosity coefficients on the basis of the measurement of the force on the sphere is discussed.

Keywords: nematic liquid crystal, Leslie-Ericksen equations, hydrodynamics

I. INTRODUCTION

There are only a few studies on the hydrodynamics of nematic liquid crystals. This is due to the complex structure of the basic hydrodynamic equations for nematic liquid crystals, the Leslie-Ericksen equations.[1,2] Besides the usual quantities of hydrodynamics, the rotation of the director, which is a unit vector parallel to the symmetry axis of the nematic liquid crystal, is taken into account in the Leslie-Ericksen equations.

There are two possibilities for experimental and theoretical studies on the flow patterns, the pressures, and the viscous forces. At first, no external torques are exerted on the director and its orientation results from viscous torques, elastic torques, and the alignment of the director at the surfaces of the sample vessel. The theoretical study of such a problem is very difficult and there are, in fact, only studies on simple shear flow between parallel plates,[1,3] on Couette flow,[4,5] and on Poiseuille flow.[6] The second possibility is an alignment of the director parallel to an electric or magnetic field of sufficient strength. The theory becomes considerably simpler as the equations for the director orientation have not to be taken into account. The remaining problems are due to the anisotropy of the nematic liquid crystal, the viscous properties of which are described by five independent viscosity coefficients (Leslie coefficients).

Our aim is the understanding of the flow of a nematic liquid crystal around a sphere. Besides a general interest in this problem, an equation for the force on the sphere can lead to a simplification for the measurement of the viscosity coefficients which is rather difficult today.

We have begun this study with the investigation of the two-dimensional analogue[7] (paper I): the flow of a nematic liquid crystal perpendicular to the axis of an infinite long cylinder. Flow patterns, pressures, and forces were calculated by means of a numerical method.

This paper contains the investigation of the three-dimensional flow of a nematic liquid crystal around a sphere with a director alignment parallel to the uniform flow at infinite distance. Due to the axial symmetry, the problem is, in fact, a two-dimensional one. It turned out that this case can be studied with analytical methods if the usually small Leslie coefficient α_1 is neglected. This paper contains the development of a series representation for the streamfunction, the velocities and the pressure and a simple formula for the force on the sphere i.e. the analogue to the Stokes formula. As the Leslie-Ericksen equations are linear, the streamfunction for this axisymmetrical case and the yet unknown streamfunction for the perpendicular director alignment can be superposed to give the general solution for the streamfunction under the assumption of a fixed director orientation.

II. HYDRODYNAMIC EQUATIONS

The basic equations are derived from the Leslie-Ericksen theory using the notation of Clark and Leslie.[8] As in paper I, we assume a fixed director orientation ($\boldsymbol{n}$ = const) and a weak anchoring of the director at the surface of the sphere. The Leslie-Ericksen equations for the steady flow of an incompressible nematic liquid crystal at low Reynolds numbers may be written as

$$\sigma_{ij,j} = 0 \tag{1}$$

$$\sigma_{ij} = -p\delta_{ij} + \alpha_1 n_k n_p A_{kp} n_i n_j + \alpha_2 N_i n_j + \alpha_3 N_j n_i + \alpha_4 A_{ij} + \alpha_5 A_{ik} n_k n_j + \alpha_6 A_{jk} n_k n_i \tag{2}$$

$$A_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \tag{3}$$

$$N_i = \tfrac{1}{2}(v_{j,i} - v_{i,j}) n_j \tag{4}$$

The usual summation convention is used and a comma preceeding a suffix denotes partial differentiation with respect to the following coordinate. The α_i's are the Leslie coefficients of the nematic liquid crystal. The A_{ij}'s are the components of the symmetric part of the velocity gradient tensor. N is the rotation of the director with respect to the fluid. p describes the dynamic pressure, i.e. the influence of the hydrostatic pressure is not taken into account. The stress tensor components σ_{ij} for a fixed director orientation parallel to the z axis are:

$$
\begin{aligned}
\sigma_{xx} &= -p + \alpha_4 v_{x,x} \\
\sigma_{xy} &= \tfrac{1}{2}\alpha_4(v_{x,y} + v_{y,x}) \\
\sigma_{xz} &= \tfrac{1}{2}(\alpha_2 + \alpha_4 + \alpha_5)v_{z,x} + \tfrac{1}{2}(-\alpha_2 + \alpha_4 + \alpha_5)v_{x,z} \\
\sigma_{yx} &= \tfrac{1}{2}\alpha_4(v_{x,y} + v_{y,x}) \\
\sigma_{yy} &= -p + \alpha_4 v_{y,y} \\
\sigma_{yz} &= \tfrac{1}{2}(\alpha_2 + \alpha_4 + \alpha_5)v_{z,y} + \tfrac{1}{2}(-\alpha_2 + \alpha_4 + \alpha_5)v_{y,z} \\
\sigma_{zx} &= \tfrac{1}{2}(\alpha_3 + \alpha_4 + \alpha_6)v_{z,x} + \tfrac{1}{2}(-\alpha_3 + \alpha_4 + \alpha_6)v_{x,z} \\
\sigma_{zy} &= \tfrac{1}{2}(\alpha_3 + \alpha_4 + \alpha_6)v_{z,y} + \tfrac{1}{2}(-\alpha_3 + \alpha_4 + \alpha_6)v_{y,z} \\
\sigma_{zz} &= -p + (\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6)v_{z,z}
\end{aligned}
\tag{5}
$$

Insertion in Equation (1) and employing the continuity equation

$$v_{x,x} + v_{y,y} + v_{z,z} = 0 \tag{6}$$

yields

$$
\begin{aligned}
&-p_{,x} + \tfrac{1}{2}\alpha_4\Delta v_x + \tfrac{1}{2}(\alpha_2 + \alpha_5)v_{z,xz} + \tfrac{1}{2}(-\alpha_2 + \alpha_5)v_{x,zz} = 0 \\
&-p_{,y} + \tfrac{1}{2}\alpha_4\Delta v_y + \tfrac{1}{2}(\alpha_2 + \alpha_5)v_{z,yz} + \tfrac{1}{2}(-\alpha_2 + \alpha_5)v_{y,zz} = 0 \\
&-p_{,z} + \tfrac{1}{2}(\alpha_3 + \alpha_4 + \alpha_6)\Delta v_z + (\alpha_1 + \alpha_5)v_{z,zz} = 0
\end{aligned}
\tag{7}
$$

where Δ is the Laplace operator. Furthermore, we introduce the modified pressure

$$p' = p - \alpha_5 v_{z,z} \tag{8}$$

and the shear viscosity coefficients

$$
\begin{aligned}
\eta_1 &= \tfrac{1}{2}(-\alpha_2 + \alpha_4 + \alpha_5) \\
\eta_2 &= \tfrac{1}{2}(\alpha_3 + \alpha_4 + \alpha_6) \\
\eta_3 &= \tfrac{1}{2}\alpha_4
\end{aligned}
\tag{9}
$$

which describe the effective viscosity for $\boldsymbol{n} \parallel \mathrm{grad}\, v$ (η_1), $\boldsymbol{n} \parallel v$ (η_2), and $\boldsymbol{n} \perp \mathrm{grad}\, v$, $\boldsymbol{n} \perp v$ (η_3).

$$
\begin{aligned}
&-p'_{,x} + \eta_3\Delta v_x + (\eta_1 - \eta_3)(v_{x,zz} - v_{z,zx}) = 0 \\
&-p'_{,y} + \eta_3\Delta v_y + (\eta_1 - \eta_3)(v_{y,zz} - v_{z,zy}) = 0 \\
&-p'_{,z} + \eta_2\Delta v_z + \alpha_1 v_{z,zz} = 0
\end{aligned}
\tag{10}
$$

These equations show that the stream line pattern and the modified pressure are completely determined by η_1, η_2, η_3, and α_1, whereas the real pressure additionally depends on α_5.

The equations are transformed into cylindrical coordinates (ρ, z). This is suggested by the director orientation and the symmetry of the boundary conditions.

$$-p'_{,\rho} + \eta_1[v_{\rho,\rho\rho} + v_{\rho,zz} + v_{\rho,\rho}/\rho - v_\rho/\rho^2] = 0 \tag{11}$$

$$-p'_{,z} + \eta_2[v_{z,\rho\rho} + (1 + \alpha_1/\eta_2)v_{z,zz} + v_{z,\rho}/\rho] = 0 \tag{12}$$

$$v_{z,z} + v_{\rho,\rho} + v_\rho/\rho = 0 \tag{13}$$

We now introduce the streamfunction Ψ which is defined by the equations

$$\Psi_{,\rho} = -\rho v_z; \qquad \Psi_{,z} = \rho v_\rho \tag{14}$$

Therefore, the continuity equation is always fulfilled. In order to get a simple expression, the usually small Leslie coefficient α_1 is neglected. In this case, Equation (11) and (12) become

$$\left(\frac{\eta_1}{\eta_2}\partial_z^2 + \partial_\rho^2 + \frac{\partial_\rho}{\rho} - \frac{1}{\rho^2}\right)\left(\partial_z^2 + \partial_\rho^2 + \frac{\partial_\rho}{\rho} - \frac{1}{\rho^2}\right)\frac{\Psi}{\rho} = 0 \tag{15}$$

This equation shows that the stream line pattern only depends in the axisymmetrical case on the ratio η_1/η_2 besides the dependence on α_1/η_2 which has been neglected.

III. STREAMFUNCTION, VELOCITIES AND PRESSURE

The solution for the streamfunction for a moving sphere is developed in Appendix A. Equation (16) is a presentation in spherical coordinates ϑ and r.

$$\Psi = \frac{\sin^2\vartheta}{4}\left(\frac{1}{r} - 3r\right) + \sin\vartheta \sum_{\substack{K=3\\ K \text{ odd}}}^{\infty} \sum_{\substack{L=1\\ L \text{ odd}}}^{K} M_{KL}\, P_K^1(\cos\vartheta)(r^{-K} - r^{-L+2}) \tag{16}$$

Dimensionless quantities are used in this and the following equations:

$$r/R \rightarrow r, \quad v/v_0 \rightarrow v, \quad \Psi/R^2 v_0 \rightarrow \Psi, \quad p'/\eta_2 v_0 R^{-1} \rightarrow p'$$

where v_0 is the velocity and R the radius of the sphere. The first term in the equation is the isotropic solution. The double sum over the products of associated Legendre polynomials of cos ϑ and powers of reciprocal radii is the anisotropic contribution. The coefficients M_{KL} which depend on the anisotropy ratio

$$\gamma = (\eta_2 - \eta_1)/\eta_1 \tag{17}$$

are calculated by the following procedure. At first, the integrals

$$I_{KL} = \frac{2K+1}{K(K+1)(K+L-1)} \int_0^1 S_L^1(x)\, P_K^1(x)\mathrm{d}x \tag{18}$$

where

$$S_L^1(x) = \frac{P_L^1(\sqrt{1+\gamma}\cdot x/\sqrt{1+\gamma x^2})}{\sqrt{1+\gamma x^2}^{\,L+1}} \tag{19}$$

have to be calculated by a numerical integration. If $\gamma < 0$, the substitution $x = (u - 1)^L + 1$ leads to a better convergence of the numerical integration. The coefficients M_{KL} are calculated by

$$M_{KL} = -\frac{I_{KL}}{(K-L+2)\cdot I_{LL}} \sum_{\substack{N=1 \\ N \text{ odd}}}^{L-2} M_{LN}(L-N+2) \tag{20}$$

with the starting value

$$M_{K1} = -\frac{3\, I_{K1}}{2(K+1)\cdot I_{11}} \tag{21}$$

Firstly, we want to discuss the convergence of the sum in Equation (16). For $\gamma = 0$, i.e. an isotropic fluid, all integrals $K \neq L$ disappear because of the orthogonality of the associated Legendre polynomials ($S_L^1(x) = P_L^1(x)$ for $\gamma = 0$). Therefore, only M_{11} is different from zero and the sum in Equation (16) does not contain this term.

The contribution of the anisotropic part increases with increasing anisotropy $|\gamma|$. For high L values and high anisotropies $|\gamma|$, M_{KL} passes through a maximum as a function of K. However, numerical stability is always achieved by enclosing sufficiently high indices K. The form of the streamlines is mainly determined by the continuity equation. In order to show the dependence on the viscosity coefficient ratio η_2/η_1, large anisotropies have to be chosen. Figures 1 to 4 show the stream line patterns for $\eta_2/\eta_1 = 10$ and 0.1 for the moving sphere and the moving fluid. Such large anisotropies are not unusual (see paper I). The stream function for the moving liquid is derived from that for the moving sphere by a change in the isotropic part as for an isotropic liquid.

The form of the stream line patterns can be understood by means of a minimum principle (see paper I). Large regions with strong velocity gradients and high effective viscosity coefficients

$$\eta_{\text{eff}} = \eta_1 \sin^2\theta + \eta_2 \cos^2\theta \tag{22}$$

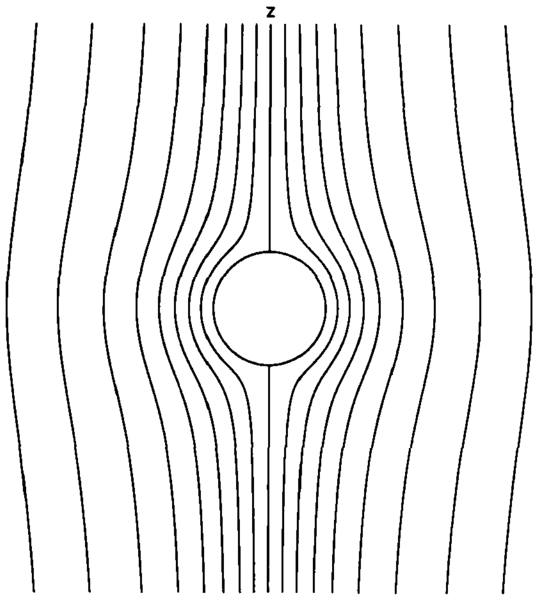

FIGURE 1 Streamline pattern for the flow of a nematic liquid crystal with the viscosity coefficient ratio $\eta_2/\eta_1 = 10$ ($\gamma = 9$) around a fixed sphere. The director orientation is parallel to the z axis in all figures.

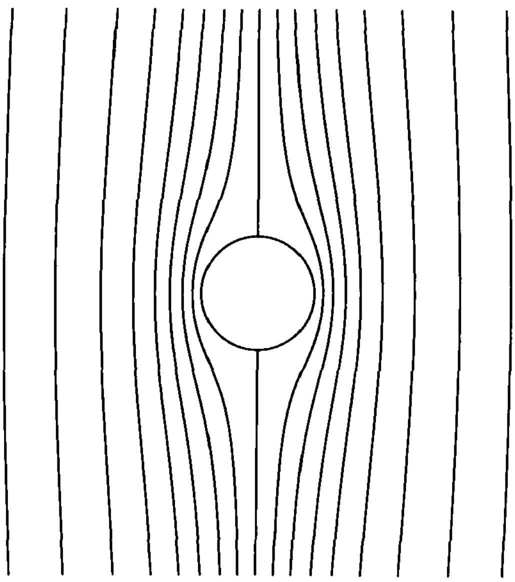

FIGURE 2 Streamline pattern for $\eta_2/\eta_1 = 0.1$ ($\gamma = -0.9$). The values for the streamfunction are the same as for Figure 1.

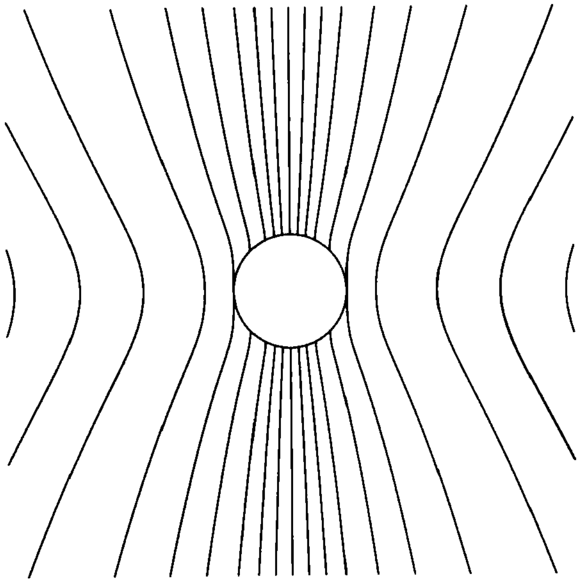

FIGURE 3 Streamline pattern for the flow of a nematic liquid crystal with the viscosity coefficient ratio $\eta_2/\eta_1 = 10$ around a moving sphere.

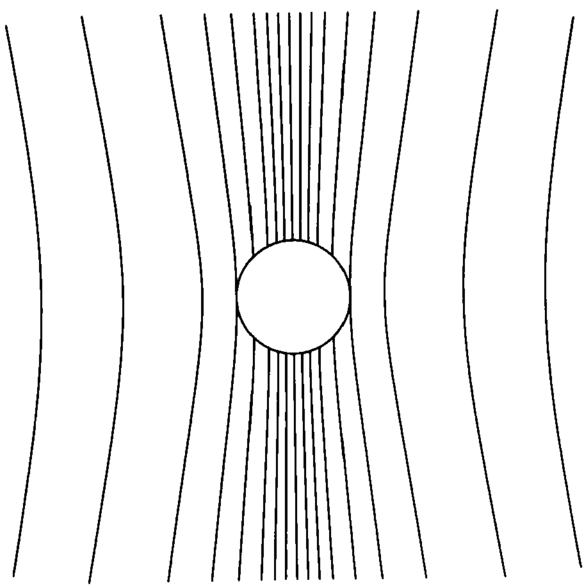

FIGURE 4 Streamline pattern for the flow of a nematic liquid crystal with the viscosity coefficient ratio $\eta_2/\eta_1 = 0.1$ around a moving sphere.

are avoided where θ is the angle between director and flow velocity. This leads to the dense stream line pattern at $\vartheta = 0$ and $\pi/2$ for $\eta_2/\eta_1 = 10$ (Figure 1) and to the wide one for $\eta_2/\eta_1 = 0.1$ (Figure 2).

The equations for the velocities v_ϑ and v_r are somewhat simpler than that for v_ρ and v_z. We present, therefore, v_ϑ and v_r which are obtained by

$$v_\vartheta = \frac{1}{\rho}\,\Psi_{,r}; \qquad v_r = -\frac{1}{r\rho}\,\Psi_{,\vartheta} \tag{23}$$

$$v_\vartheta = -\frac{\sin\vartheta}{4}\left(\frac{1}{r^3} + \frac{3}{r}\right) + \sum_{K=3}^{\infty}\sum_{L=1}^{K} M_{KL}\, P_K^1(\cos\vartheta)\,[(L-2)r^{-L} - Kr^{-K-2}] \tag{24}$$

$$v_r = -\frac{\cos\vartheta}{2}\left(\frac{1}{r^3} - \frac{3}{r}\right) + \sum_{K=3}^{\infty}\sum_{L=1}^{K} M_{KL}\, K(K+1)\, P_K(\cos\vartheta)\,(r^{-K-2} - r^{-L}) \tag{25}$$

The pressure is determined by an integration of the differential Equation (12) for v_z assuming $\alpha_1 = 0$ (see Appendix B).

$$p' = \frac{3}{2}\frac{\cos\vartheta}{r^2} + \int_1^{\cos\vartheta} \sum_{K=3}^{\infty}\sum_{L=1}^{K} M_{KL}\, \frac{K(K+1)\,(K-L+2)(K+L-1)}{(2K+1)\,r^2\sin^2\vartheta}\left[\frac{\sqrt{1-x^2}}{r\sin\vartheta}\right]^{L-1} \cdot[(K+L+1)P_{K+1}(x) + (K-L)P_{K-1}(x)]\,dx \tag{26}$$

As the products $\sqrt{1-x^2}^{\,L-1}\cdot P_N(x)$ are polynomials in x ($L-1$ is even), the integrals can be calculated by analytical methods. The resulting four-fold sum is so complicated that we have evaluated the pressure with Equation (26) using a numerical integration. The summation and most of the factors in the terms in the sum can be extracted from the integral as r as well as ϑ are constants with respect to the integration. Nevertheless, we performed the numerical integration in the form presented above as the error determination of the integration is simpler in this form. The lower integration limit 1 can be replaced by 0. We have performed, therefore, the integration between 0 and cos ϑ for small cos ϑ values and vice versa. For $\vartheta = 0$, the improper integral can be calculated with analytical methods (see Appendix B).

$$p'_{\vartheta=0} = \frac{3}{2}r^{-2} - \sum_{K=3}^{\infty}\sum_{L=1}^{K} M_{KL}\, \frac{K(K+1)(K-L+2)(K+L-1)}{L+1}\, r^{-L-1} \tag{27}$$

Figure 5 shows the modified pressure p' on the sphere for several anisotropy ratios of the viscosity coefficients. The modified pressure is high in regions where the velocity gradient is high, i.e. at $\vartheta = 0$ and 180° for $\gamma \gg 0$ and at $\vartheta = 90°$ for $\gamma \ll 0$. For the determination of the real pressure p, the term $\alpha_5 v_{z,z}$ has to be added. $v_{z,z}$ can be calculated from Equation (B1), but the resulting expression is

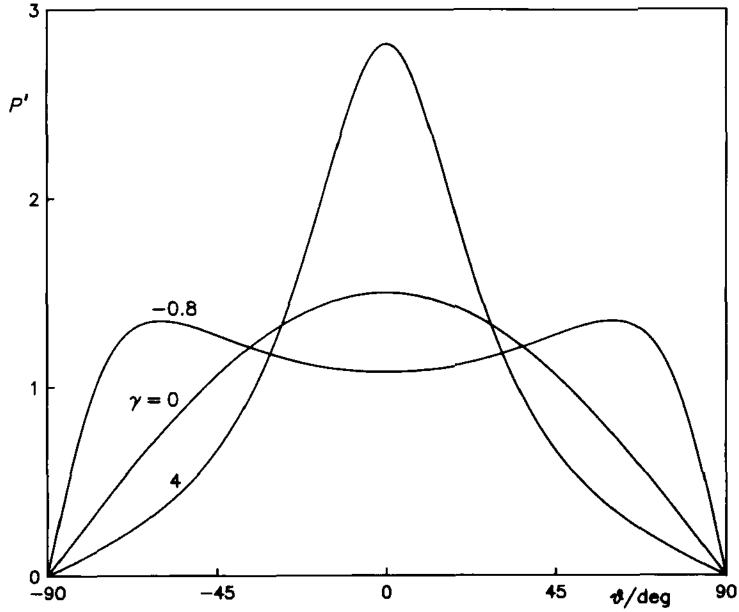

FIGURE 5 Dimensionless modified pressure p' at the surface of the moving sphere as a function of the polar angle ϑ for different anisotropy ratios $\gamma = (\eta_2 - \eta_1)/\eta_1$.

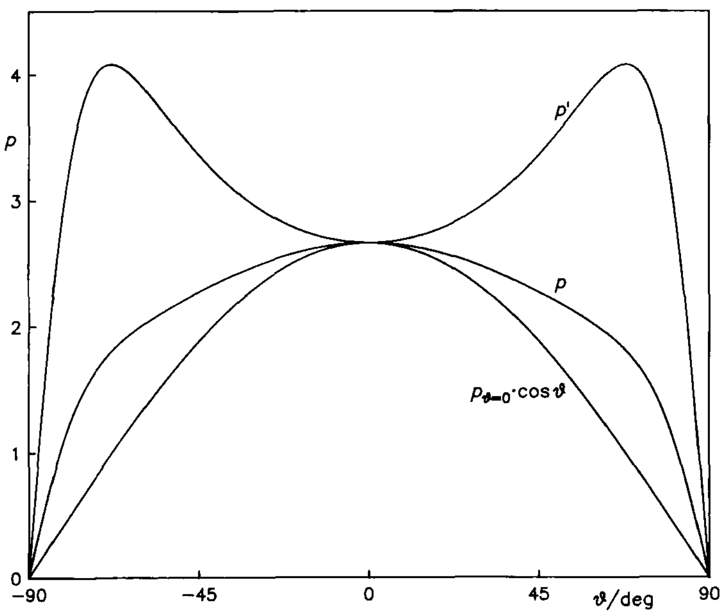

FIGURE 6 Dimensionless pressures p and p' at the surface of the moving sphere as a function of the polar angle ϑ for the Nematic Phase IV at 20°C. For a comparison, the function $p(\vartheta = 0) \cdot \cos \vartheta$ is added.

rather lengthy. The calculation of $v_{z,z}$ on the surface of the sphere is, however, simple:

$$v_{z,z} = -\sin\vartheta \cos\vartheta \, v_{\vartheta,r} \tag{28}$$

which leads to

$$p = p' - \frac{\alpha_5}{\eta_2} \sin\vartheta \cos\vartheta \, v_{\vartheta,r} \tag{29}$$

in dimensionless form. Figure 6 shows the real, dimensionless pressure for the eutectic mixture of the two isomeric 4-methoxy-4′-*n*-butylazoxybenzenes (Nematic Phase IV, Merck AG) at 20°C using the viscosity data of Kneppe *et al.*[9] (η_1 = 0.227, η_2 = 0.0275, and α_5 = 0.147 Pa·s). The influence of α_1 is neglected. In contrast to the modified pressure, no maxima are observed for the real pressure. However, a strong enhancement of the pressure in comparison to a cos function remains in this region.

IV. FORCE ON THE SPHERE

The z component of the force differential is obtained by

$$dF_z = \sigma_{zx} dS_x + \sigma_{zy} dS_y + \sigma_{zz} dS_z \tag{30}$$

where the dS_i are the surface elements on the sphere. As dS_r is the only non-vanishing component on the sphere, introduction of dS_r leads to

$$dF_z = (\sin\vartheta \cos\phi \, \sigma_{zx} + \sin\vartheta \sin\phi \, \sigma_{zy} + \cos\vartheta \, \sigma_{zz}) \, dS_r \tag{31}$$

Using the tensor components given in Equation (5) and substitution of the cartesian velocity components by spherical components results in the dimensionless equation ($F/\eta_2 R v_0 \rightarrow F$)

$$dF_z = -(\cos\vartheta \, p' + (1 + \frac{\alpha_1}{\eta_2} \cos^2\vartheta) \sin\vartheta \, v_{\vartheta,r}) \sin\vartheta \, d\vartheta \, d\phi \tag{32}$$

Integration gives

$$F_z = -4\pi \int_0^{\pi/2} (\cos\vartheta \, p' + (1 + \frac{\alpha_1}{\eta_2} \cos^2\vartheta) \sin\vartheta \, v_{\vartheta,r}) \sin\vartheta \, d\vartheta \tag{33}$$

Neglection of the α_1 term and integration over ϑ, which is described in Appendix C, leads in dimensional form to

$$F_z = -4\pi \sqrt{\eta_1 \eta_2} \, R v_0 \frac{\gamma \sqrt{|\gamma|}}{\sqrt{|\gamma|(1+\gamma)} - \mathrm{acos}\sqrt{1+\gamma}}$$

where

$$\mathrm{acos}\sqrt{1+\gamma} = \begin{cases} \mathrm{Arch}\sqrt{1+\gamma} & \text{for } \gamma > 0 \\ \arccos\sqrt{1+\gamma} & \text{for } \gamma < 0 \end{cases} \tag{34}$$

The isotropic case $\gamma = 0$ gives an undeterminated fraction but the limit for $\gamma \rightarrow 0$ is the Stokes formula.

Figure 7 shows the dimensionless force

$$F_z^\star = \frac{-F_z}{6\pi\sqrt{\eta_1\eta_2}\, Rv_0} \tag{35}$$

as a function of $\sqrt{\eta_2/\eta_1}$ which gives nearly a straight line. The point of intersection with the ordinate axis is $4/3\pi$.

The influence of α_1 on the force is taken into account as a first-order pertubation which is described in Appendix C. The result in dimensional form is

$$F_z = -4\pi\sqrt{\eta_1\eta_2}\, Rv_0 \left(\frac{\gamma\sqrt{|\gamma|}}{\sqrt{|\gamma|(1+\gamma)} - \mathrm{acos}\sqrt{1+\gamma}} + \frac{\alpha_1}{\sqrt{\eta_1\eta_2}} \left(\frac{1}{\gamma} + \frac{1}{3} + \frac{2\ \mathrm{asin}\sqrt{|\gamma|}}{3(\mathrm{asin}\sqrt{|\gamma|} - \sqrt{|\gamma|(1+\gamma)})} \right) \right) \tag{36}$$

where the definition of asin $\sqrt{|\gamma|}$ corresponds to that given in Equation (34). $\gamma = 0$ leads to

$$F_z = -6\pi\eta Rv_0(1 + \frac{2}{15}\frac{\alpha_1}{\eta}) \tag{37}$$

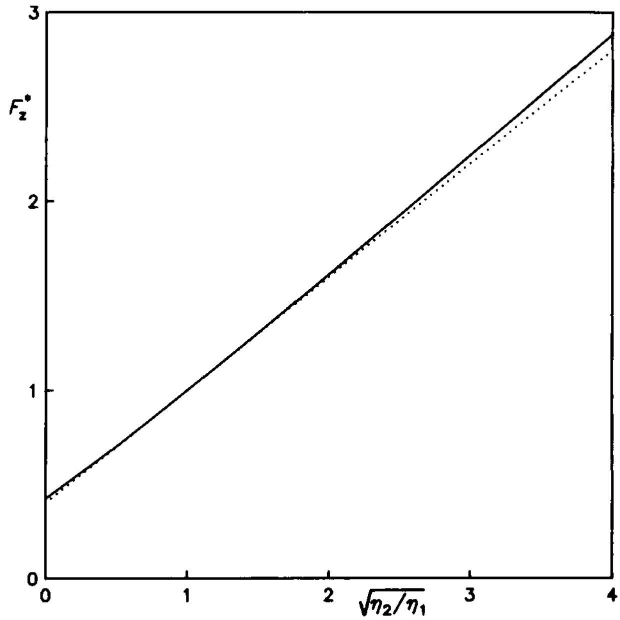

FIGURE 7 Dimensionless force F_z^* as a function of the square root of the viscosity coefficient ratio η_2/η_1. The tangent (dotted line) at the isotropic point with a slope of 3/5 is added as a guide for the eye.

This shows the small influence of α_1 which itself is usually small in comparison to the mean viscosity coefficients. There are only a few liquid crystals for which α_1 has been measured.[10,11] In these cases, Equation (36) leads to a correction of 5 – 10%.

The determination of viscosity coefficients or Leslie coefficients by a measurement of the force on the sphere in the axisymmetrical case is not possible as there is only one measurement for two unknown quantities (or three if α_1 is taken into account). A measurement with the director perpendicular to the movement of the sphere gives an additional information, but the force depends on η_1, η_2, η_3, and α_1 as will be shown in a forthcoming paper. Measurements under oblique angles do not lead to additional informations. We hope that it will be possible to calculate and to measure the force on prolate and oblate ellipsoids in the axisymmetrical and the perpendicular case. This should allow the determination of all viscosity coefficients. A further problem, which has to be taken into account in the calculations, is the influence of the walls of the sample vessel.

The fixed director orientation parallel to the z axis can be accomplished by application of a magnetic field of sufficient strength. The surface alignment at the sphere and the torque on the director due to the shear flow lead to deviations of the director orientation from the field direction. The influence of both effects, which have already been discussed in paper I, can be neglected if the field strength is high enough.

APPENDIX A

Dimensionless quantities are used in Appendices A, B, and C:

$$r/R \rightarrow r, \quad \rho/R \rightarrow \rho, \quad v/v_0 \rightarrow v, \quad \Psi/R^2 v_0 \rightarrow \Psi, \quad p/\eta_2 v_0 R^{-1} \rightarrow p$$

Equation (15), which reads identically in dimensionless quantities, allows to calculate the streamfunction for a moving sphere as well as for a moving liquid. The asymptotic behaviour of the stream function for a nematic liquid crystal for ρ or $r \rightarrow \infty$ agrees with that for an isotropic fluid. For the moving sphere, the term with the highest power in ρ is of the form $\rho \cdot \sin \vartheta$ and Ψ/ρ remains finite for $\rho \rightarrow \infty$ which is advantageous for an expansion to be performed later on. Therefore, the stream function is calculated for the moving sphere in the following.

Equation (15) can be written as a system of two coupled differential equations

$$\left(\frac{\eta_1}{\eta_2}\,\partial_z^2 + \partial_\rho^2 + \frac{\partial_\rho}{\rho} - \frac{1}{\rho^2}\right) \Lambda(\rho,z) = 0 \tag{A1}$$

$$\left(\partial_z^2 + \partial_\rho^2 + \frac{\partial_\rho}{\rho} - \frac{1}{\rho^2}\right) \frac{\Psi}{\rho} = \Lambda(\rho,z) \tag{A2}$$

Equation (A1) can be solved by the following method. A scaling of the z axis by

$$\tilde{z} = \sqrt{\eta_2/\eta_1} \cdot z \tag{A3}$$

leads to the differential equation

$$(\partial_{\tilde{z}}^2 + \partial_\rho^2 + \frac{\partial_\rho}{\rho} - \frac{1}{\rho^2})\, \Lambda(\rho,\tilde{z}) = 0 \tag{A4}$$

which is transformed into spherical coordinates $\tilde{\vartheta}$, $\tilde{r}$

$$(\tilde{r}\partial_{\tilde{r}}^2\, \tilde{r} + \partial_{\tilde{\vartheta}}^2 + \cot\tilde{\vartheta}\, \partial_{\tilde{\vartheta}} - \sin^{-2}\tilde{\vartheta})\, \Lambda = 0 \tag{A5}$$

This equation has the general solution

$$\Lambda = \sum_{L=1}^{\infty} A_L \frac{P_L^1(\cos\tilde{\vartheta})}{\tilde{r}^{L+1}} \tag{A6}$$

where solutions with positive powers of $\tilde{r}$ ($\tilde{r}^L\, P_L^1(\cos\tilde{\vartheta})$) are not taken into account as they lead to positive powers in the radius for Ψ/ρ. Transformation back to the original spherical coordinate system gives

$$\Lambda = \sum_{L=1}^{\infty} A_L \frac{S_L^1(\cos\vartheta)}{r^{L+1}} \tag{A7}$$

with

$$S_L^1(x) = \frac{P_L^1(\sqrt{1+\gamma}\cdot x/\sqrt{1+\gamma x^2})}{\sqrt{1+\gamma x^2}^{L+1}} \tag{A8}$$

The right hand side of the inhomogenous differential Equation (A2) is now known. The operator of this differential equation can be traced back to the Laplace operator Δ in cylindrical coordinates

$$\Delta(e^{i\phi}\frac{\Psi}{\rho}) = e^{i\phi}\cdot(\partial_z^2 + \partial_\rho^2 + \frac{\partial_\rho}{\rho} - \frac{1}{\rho^2})\frac{\Psi}{\rho} = e^{i\phi}\cdot\Lambda \tag{A9}$$

The solution is the Poisson integral

$$e^{i\phi}\frac{\Psi}{\rho} = -\frac{1}{4\pi}\int_V \frac{e^{i\phi'}\Lambda(\cos\vartheta',r')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d^3r' \tag{A10}$$

$1/|r - r'|$ is expanded in spherical harmonics Y_{KM}.

$$\frac{\Psi}{\rho} = -e^{-i\phi}\int_V e^{i\phi'}\Lambda(\cos\vartheta',r')\sum_{K=0}^{\infty}\sum_{M=-K}^{K}\frac{1}{2K+1}\frac{r_<^K}{r_>^{K+1}}Y^{\star}_{KM}(\cos\vartheta',\phi')\,Y_{KM}(\cos\vartheta,\phi)$$

$$\cdot\sin\vartheta' r'^2 dr'\, d\vartheta'\, d\phi'$$

$$= -\frac{e^{-i\phi}}{4\pi}\sum_{L=1}^{\infty}\sum_{K=0}^{\infty}\sum_{M=-K}^{K} A_L P_K^M(\cos\vartheta)\, e^{iM\phi}\frac{(K-M)!}{(K+M)!}\int_1^{\infty}\frac{r'^2\, r_<^K}{r_>^{K+1} r'^{L+1}}\,dr'$$

$$\cdot\int_0^{\pi} S_L^1(\cos\vartheta')\, P_K^M(\cos\vartheta')\sin\vartheta'\, d\vartheta' \int_0^{2\pi} e^{-iM\phi'}e^{i\phi'}\, d\phi' \tag{A11}$$

The integral over ϕ' gives $2\pi\delta_{M1}$. The integral over r' gives

$$\frac{1}{K-L+2}\left(\frac{2K+1}{K+L-1}\, r^{-L+1} - r^{-K-1}\right) \tag{A12}$$

with the condition

$$K+L > 1 \quad \text{and} \quad K+2 \neq L \tag{A13}$$

In the integral over ϑ', $\cos\vartheta'$ is substituted by x. Furthermore, the solution of the homogenous equation to Equation (A2) has to be added, whereby only terms with negative powers in the radius are taken into account. The calculation of Ψ finally leads to

$$\Psi = \sum_{L=1}^{\infty}\sum_{K=1}^{\infty}\frac{A_L \sin\vartheta\, P_K^1(\cos\vartheta)}{2K(K+1)(K-L+2)}\left(r^{-K} - \frac{2K+1}{K+L-1}\, r^{-L+2}\right)\int_{-1}^{1} S_L^1(x)\, P_K^1(x)\,dx$$

$$+\sum_{K=1}^{\infty} B_K \sin\vartheta\, P_K^1(\cos\vartheta)\, r^{-K} \tag{A14}$$

The symmetry of the streamfunction requires $K = odd$ and the symmetry of the integrand leads to $L = odd$. One can prove by analytical methods that the integrals vanish for $L > K$ for odd K and L values.

Introduction of

$$I_{KL} = \frac{2K+1}{K(K+1)(K+L-1)}\int_0^1 S_L^1(x) P_K^1(x)\,dx \tag{A15}$$

leads to

$$\Psi = \sum_{K=1}^{\infty}\sum_{L=1}^{K} \frac{A_L \sin\vartheta \; P_K^1(\cos\vartheta)\; I_{KL}}{K-L+2}\left(r^{-K}\frac{K+L-1}{2K+1} - r^{-L+2}\right) + \sum_{K=1}^{\infty} B_K \sin\vartheta \; P_K^1(\cos\vartheta)\; r^{-K} \tag{A16}$$

The boundary conditions for the stream function are the same as for an isotropic liquid.

$$\text{I)}\quad \Psi\big|_{r=1} = -\tfrac{1}{2}\sin^2\vartheta, \qquad\qquad \text{II)}\quad \Psi_{,r}\big|_{r=1} = -\sin^2\vartheta$$

The first condition yields

$$B_1 = \frac{1}{2} + \frac{1}{3}\sum_{L=1}^{\infty} A_L I_{1L}, \qquad B_K = \sum_{L=1}^{\infty} \frac{A_L I_{KL}}{2K+1} \tag{A17}$$

Insertion in Equation (A16) gives

$$\Psi = \frac{\sin^2\vartheta}{2}\left[2M_{11}(r - r^{-1}) - r^{-1}\right] + \sin\vartheta \sum_{K=3}^{\infty}\sum_{L=1}^{K} M_{KL} P_K^1(\cos\vartheta)\left(r^{-K} - r^{-L+2}\right) \tag{A18}$$

with

$$M_{KL} = \frac{A_L I_{KL}}{K-L+2} \tag{A19}$$

The second condition gives

$$M_{K1} = -\frac{3\, I_{K1}}{2(K+1)\cdot I_{11}}, \qquad M_{KL} = \frac{I_{KL}}{(L-K-2)I_{LL}} \sum_{\substack{N=1 \\ N \text{ odd}}}^{L-2} M_{LN}(L-N+2) \tag{A20}$$

which leads to

$$\Psi = \frac{\sin^2\vartheta}{4}\left(\frac{1}{r} - 3r\right) + \sin\vartheta \sum_{K=3}^{\infty}\sum_{L=1}^{K} M_{KL}\; P_K^1(\cos\vartheta)\left(r^{-K} - r^{-L+2}\right) \tag{A21}$$

with odd indices K and L.

APPENDIX B

The calculation of the pressure can be performed by an integration of Equation (11) or (12). v_z is calculated from Equation (24) and (25) and becomes after some manipulations on the Legendre polynomials

$$v_z = \frac{3}{4}\cos^2\vartheta\left(\frac{1}{r}-\frac{1}{r^3}\right)+\frac{1}{4}\left(\frac{3}{r}+\frac{1}{r^3}\right)+\sum_{K=3}^{\infty}\sum_{L=1}^{K}M_{KL}(K+1)$$
$$\cdot\left[K\,P_{K+1}(\cos\vartheta)\,r^{-K-2}-((K-L+2)\cos\vartheta\,P_K(\cos\vartheta)+(L-2)\,P_{K+1}(\cos\vartheta))\,r^{-L}\right] \tag{B1}$$

As v_z contains only (non-associated) Legendre polynomials and the differential Equation (12) is somewhat shorter, we have used this equation for the integration

$$p' = \int_{z'=\infty}^{z}(v_{z,\rho\rho}+v_{z,z'z'}+v_{z,\rho}/\rho)\,dz'+f(\rho) \tag{B2}$$

where the integration has to be performed for ρ = const. The arbitrary function $f(\rho)$ can not fulfill the necessary symmetry for the pressure and must, therefore, vanish. As v_z is given in spherical coordinates (r', ϑ'), the differentials were transformed into this system

$$p' = \int_{z'=\infty}^{z}\left[r'^{-1}\frac{\partial^2(r'v_z)}{\partial r'^2}-\frac{2\cos\vartheta'}{r'^2}\frac{\partial v_z}{\partial\cos\vartheta'}+\frac{\sin^2\vartheta'}{r'^2}\frac{\partial^2 v_z}{\partial\cos\vartheta'^2}\right]dz' \tag{B3}$$

which after a long calculation gives

$$p' = \int_{z'=\infty}^{z}\Big[\frac{3(1-3\cos^2\vartheta')}{2r'^3}+\sum_{K=3}^{\infty}\sum_{L=1}^{K}M_{KL}\frac{K(K+1)(K-L+2)(K+L-1)}{2K+1}r'^{-L-2}$$
$$\cdot[(K+L+1)P_{K+1}(\cos\vartheta')+(K-L)P_{K-1}(\cos\vartheta')]\Big]\,dz' \tag{B4}$$

where r' and ϑ' have to be represented as functions of ρ and z'. The substitution

$$x=\cos\vartheta'=\frac{z'}{\sqrt{\rho^2+z'^2}} \tag{B5}$$

finally leads to

$$p' = \frac{3}{2}\frac{\cos\vartheta}{r^2}+\int_{1}^{\cos\vartheta}\sum_{K=3}^{\infty}\sum_{L=1}^{K}M_{KL}\frac{K(K+1)(K-L+2)(K+L-1)}{(2K+1)r^2\sin^2\vartheta}\left[\frac{\sqrt{1-x^2}}{r\sin\vartheta}\right]^{L-1}$$
$$\cdot[(K+L+1)P_{K+1}(x)+(K-L)P_{K-1}(x)]\,dx \tag{B6}$$

The integrals were calculated by a numerical method.

The case $\vartheta = 0$ is evaluated by means of Equation (B4) in which the Legendre polynomials are equal to 1. The integration can be performed by analytical methods leading to Equation (27) which is given in the main text.

APPENDIX C

The calculation of the integral

$$F_z = -4\pi \int_0^{\pi/2} (\cos\vartheta\, p' + \sin\vartheta\, v_{\vartheta,r}) \sin\vartheta d\vartheta \tag{C1}$$

for the force is performed as follows. Insertion of the modified pressure p' (Equation (26)), of the velocity v_ϑ (Equation (24)) for $r = 1$ and integration of the "isotropic" parts leads to

$$F_z = -6\pi\Big[1 + \frac{2}{3}\sum_{K=3}^{\infty}\sum_{L=1}^{K} M_{KL} \frac{K(K+1)(K-L+2)(K+L-1)}{2K+1} \int_0^{\pi/2} \frac{\cos\vartheta}{\sin^L\vartheta} \int_1^{\cos\vartheta} \sqrt{1-x^2}^{L-1}$$
$$\cdot((K+L+1)P_{K+1}(x) + (K-L)P_{K-1}(x))\,dx d\vartheta \tag{C2}$$
$$+ \frac{2}{3}\sum_{K=3}^{\infty}\sum_{L=1}^{K} M_{KL}(K(K+2) - L(L-2)) \int_0^{\pi/2} P_K^1(\cos\vartheta)\sin^2\vartheta d\vartheta \Big]$$

The second integral is transformed by $\cos\vartheta = y$ and gives

$$-\frac{1}{4}\frac{K(K+1)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}(3-K))\,\Gamma(\frac{1}{2}(4+K))} \tag{C3}$$

which vanishes as $3-K$ is a non-positive even integer. The first integral is also transformed by $\cos\vartheta = y$ and a partial integration over y leads for $L \neq 1$ to

$$\frac{1}{L-1}\int_0^1 (\sqrt{1-x^2}^{L-1} - 1)((K+L+1)P_{K+1}(x) + (K-L)P_{K-1}(x))\,dx \tag{C4}$$

The integrals over the Legendre polynomials (muliplied with -1) vanish. The remaining integrals over $P_{K+1}(x)$ give

$$\frac{K+L+1}{2\,(L-1)} \frac{\pi\Gamma^2(\frac{1}{2}(L+1))}{\Gamma(\frac{1}{2}(L+K+4))\,\Gamma(\frac{1}{2}(L-K))\,\Gamma(\frac{1}{2}(K+3))\,\Gamma(-\frac{1}{2}K)} \tag{C5}$$

which vanish as $L-K$ is a non-positive even integer. Correspondingly, the integrals over $P_{K-1}(x)$ vanish for $L < K$. For $K = L$, the coefficient in front of $P_{K-1}(x)$ vanishes. The only integral which has to be taken into account, therefore, is the integral which has been excluded in the calculation of Equation (C4)

$$F_z = -6\pi\left[1 + \frac{2}{3}\sum_{K=3}^{\infty} M_{K1} \frac{K^2(K+1)^2}{2K+1} \int_0^1 \frac{y}{1-y^2} \int_1^y ((K+2)P_{K+1}(x) + (K-1)P_{K-1}(x))\,dx\,dy\right] \tag{C6}$$

The integrals over x can be solved and simplified

$$F_z = -6\pi\left[1 + \frac{2}{3}\sum_{K=3}^{\infty} M_{K1} \frac{K^2(K+1)^2}{2K+1} \int_0^1 \frac{y^2}{1-y^2} (P_{K+1}(y) - P_{K-1}(y))\,dy\right] \tag{C7}$$

The difference of the Legendre polynomials can be represented by $P'_K(y)$ and a partial integration gives

$$F_z = -6\pi\left[1 - \frac{2}{3}\sum_{K=3}^{\infty} M_{K1}\, K(K+1)\,\left(1 - 2\int_0^1 y\, P_K(y)\,dy\right)\right] \tag{C8}$$

A further partial integration gives

$$F_z = -6\pi\left[1 - \frac{2}{3}\sum_{K=3}^{\infty} M_{K1}\, K(K+1)\right] = -6\pi\left(1 + \sum_{K=3}^{\infty} K \frac{I_{K1}}{I_{11}}\right) \tag{C9}$$

I_{11} and I_{K1} are replaced by the corresponding integrals (Equation (A15))

$$F_z = -6\pi\left(1 - \frac{2}{3}\sum_{K=3}^{\infty} \frac{2K+1}{K(K+1)} \frac{\int_0^1 \frac{\sqrt{1-x^2}}{\sqrt{1+\gamma x^2}^{3}} P_K^1(x)\,dx}{\int_0^1 \frac{1-x^2}{\sqrt{1+\gamma x^2}^{3}}\,dx}\right) \tag{C10}$$

Using

$$\sqrt{1-x^2}\, P_K^1(x) = \frac{K(K+1)}{2K+1}(P_{K+1}(x) - P_{K-1}(x)) \tag{C11}$$

only two terms of the numerator remain after the summation

$$\int_0^1 \frac{-P_2(x)}{\sqrt{1+\gamma x^2}^3}\,dx + \lim_{K\to\infty}\int_0^1 \frac{P_{K+1}(x)}{\sqrt{1+\gamma x^2}^3}\,dx \tag{C12}$$

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where the second integral vanishes. The remaining integral can be solved and the final result is obtained after some manipulations

$$F_z = -4\pi\sqrt{\eta_1/\eta_2}\;\frac{\gamma\sqrt{|\gamma|}}{\sqrt{|\gamma|(1+\gamma)} - \mathrm{acos}\sqrt{1+\gamma}}$$

where

$$\mathrm{acos}\sqrt{1+\gamma} = \begin{cases}\mathrm{Arch}\sqrt{1+\gamma} & \text{for } \gamma > 0\\ \arccos\sqrt{1+\gamma} & \text{for } \gamma < 0\end{cases} \tag{C13}$$

The influence of α_1 on the force is determined by means of a first order perturbation calculation. Equation (33) is now used with the α_1 term whereas the influence of α_1 on p' and v_ϑ is not taken into account. This leads to two additional terms in Equation (C2). The first term arises from the "isotropic" term of v_ϑ which becomes after integration in dimensionless form

$$-\frac{4\pi}{5}\frac{\alpha_1}{\eta_2} \tag{C14}$$

The anisotropic part gives

$$-4\pi\frac{\alpha_1}{\eta_2}\sum_{K=3}^{\infty}\sum_{L=1}^{K} M_{KL}\,(K(K+2) - L(L-2))\int_0^{\pi/2}\sin^2\vartheta\,\cos^2\vartheta\,P_K^1(\cos\vartheta)\,d\vartheta \tag{C15}$$

The integral can be solved. It vanishes for $K \geq 5$. The remaining integral for $K = 3$ gives

$$-4\pi\frac{\alpha_1}{\eta_2}\left(-\frac{8}{35}\right)\sum_{L=1}^{K} M_{3L}\,(15 - L(L-2)) \tag{C16}$$

Insertion of the integrals I_{KL} leads to

$$-4\pi\frac{\alpha_1}{\eta_2}\left(-\frac{24}{35}\right)\frac{I_{31}}{I_{11}} \tag{C17}$$

The integrals I_{31} and I_{11} are solved to give

$$-4\pi\frac{\alpha_1}{\eta_2}\left(\frac{1}{\gamma} + \frac{2}{15} + \frac{2\ \mathrm{asin}\sqrt{|\gamma|}}{3(\mathrm{asin}\sqrt{|\gamma|} - \sqrt{|\gamma|(1+\gamma)})}\right) \tag{C18}$$

where the definition of $\operatorname{asin}\sqrt{|\gamma|}$ corresponds to that of $\operatorname{acos}\sqrt{|\gamma|}$. The final result for the force becomes

$$F_z = -4\pi\sqrt{\eta_1/\eta_2}\left(\frac{\gamma\sqrt{|\gamma|}}{\sqrt{|\gamma|(1+\gamma)} - \operatorname{acos}\sqrt{1+\gamma}} + \frac{\alpha_1}{\sqrt{\eta_1\eta_2}}\left(\frac{1}{\gamma}+\frac{1}{3}+\frac{2\operatorname{asin}\sqrt{|\gamma|}}{3(\operatorname{asin}\sqrt{|\gamma|}-\sqrt{|\gamma|(1+\gamma)})}\right)\right) \tag{C19}$$

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